New Durham Sep.t 4th. 1824. (page.

Dear Sir) When I saw you at Guilford Court, a year ago, I think I said something about sending, for your inspection, the solutions of a few of my algebraical problems. — Owing to business that devolved upon me rather unexpectedly, I have had but little leisure to attend to such speculations for some time past; lately, however, I have had occasion to solve two of Bonnycastle's Diophantine Problems, and I herewith transmit to you the processes at large, together with a few other matters that may be thought useful; not doubting but you will be easily able to put the whole into a far better shape than I have done.

Diophantine Problem.

See Bonnycastle's Algebra, Philadelphia Edition, Problem 7, Page 159.

To find three square numbers such, that the sum of every two of them shall be a square number.

This has always been a popular question, and is, moreover, considered as quite curious and useful. Bonnycastle's solution of it is neat and ingenious; — his answer is 6325; 5796, and 528, which are the roots of the squares required; M.r Euler has given another answer, viz. 44, 117, and 240, which are the roots. Bonnycastle says, these latter roots are the least, in whole numbers, that have as yet been discovered, and I may add, that I am pretty confident, no smaller roots in whole numbers, than these last, will ever be found to satisfy the conditions of the problem. — Bonnycastle is indebted to Euler not only for the solution which he has given in his introduction to Algebra, but also for the most simple of the foregoing answers. I shall propose, for this question, a solution of my own, different from all which I have seen; and in order that the process may be as simple as possible, I shall employ such unknown letters only, as will be absolutely necessary for our purpose.

Let x^2, y^2 and z^2 represent the square numbers sought, then, by the conditions, $\left.\begin{array}{r} x^2+y^2 \\ x^2+z^2 \\ \text{and } y^2+z^2 \end{array}\right\}$ are to be squares.

Divide each of these by the square x^2, and we shall have $\left.\begin{array}{r} \dfrac{x^2+y^2}{x^2} = 1+\dfrac{y^2}{x^2} \\ \dfrac{x^2+z^2}{x^2} = 1+\dfrac{z^2}{x^2} \\ \text{and } \dfrac{y^2+z^2}{x^2} = \dfrac{y^2}{x^2}+\dfrac{z^2}{x^2} \end{array}\right\}$ all which are to be made $\square^s$.

Put $\frac{y}{x} = \frac{s^2-1}{2s}$, then $\frac{y^2}{x^2} = \left(\frac{s^2-1}{2s}\right)^2 = \frac{s^4-2s^2+1}{4s^2}$; whence $1+\frac{y^2}{x^2} = \frac{4s^2+s^4-2s^2+1}{4s^2} = \frac{s^4+2s^2+1}{4s^2}$, which is evidently a square.

Again put $\frac{z}{x} = \frac{s^2-4}{4s}$; then $\frac{z^2}{x^2} = \left(\frac{s^2-4}{4s}\right)^2 = \frac{s^4-8s^2+16}{16s^2}$, whence $1+\frac{z^2}{x^2} = \frac{16s^2+s^4-8s^2+16}{16s^2} = \frac{s^4+8s^2+16}{16s^2}$, which is also a square; it therefore only remains to make $\frac{y^2}{x^2}+\frac{z^2}{x^2}$ a $\square$.

To effect this, let the values of these quantities, as expressed above in terms of s, be brought to the same denominator, and added together, we shall then have

$\frac{s^4-2s^2+1}{4s^2}$, or $\frac{4s^4-8s^2+4}{16s^2}+\frac{s^4-8s^2+16}{16s^2} = \frac{5s^4-16s^2+20}{16s^2}$, which is to be made a square. Now since the denominator $16s^2$ is already a square, it may be thrown away, and then $5s^4-16s^2+20$ only remains to be made a square.

But this latter formula, while in its present shape, is inconvenient for our purpose; because neither its first term $5s^4$, nor its last term 20 is a square, it will, therefore never admit of a solution, unless we know, or can find, some satisfactory case; that is, unless we can find some value for s, that shall make $5s^4-16s^2+20$ a square. Now I find, by trial, that if $s=1$, this quantity becomes $5-16+20=9$, which is a square; but if I take 1 for the constant value of s, I shall have above $\frac{y}{x} = \frac{s^2-1}{2s} = \frac{1-1}{2} = \frac{0}{2}$, wherefore $\frac{y}{x}=0$, which is contrary to the nature of the question, for $\frac{y}{x}$ must have some value, either positive or negative.

Let us take, then, the quantity $5s^4-16s^2+20$, and change the shape of it, by substituting $q+1$ instead of s; thus we shall have $5s^4-16s^2+20 = 5q^4+20q^3+14q^2-12q+9$, which is to be made a square. Let its square root be pq^2-2q+3, then, by involution, we derive

$$5q^4+20q^3+14q^2-12q+9 = p^2q^4-4pq^3+6pq^2-12q+9, \text{ or}$$

$$5q^4+20q^3+14q^2 = p^2q^4-4pq^3+6pq^2+4q^2;$$ but dividing this last equation by q^2, which is common to all the terms, we derive $5q^2+20q+14 = p^2q^2-4pq+6p+4$; in which, by equating the homologous terms, we obtain $14 = 6p+4$, or $p=\frac{5}{3}$. Again, since the remaining terms

$$5q^2+20q = p^2q^2-4pq, \text{ or } 5q+20 = \frac{25}{9}q - \frac{20}{3},$$ we have, by reduction, $q = -\frac{240}{20} = -12$, consequently s, (or $q+1$) $= -12+1 = -11$.

Now if we substitute -11 in the place of s, we shall have above $\frac{y}{x} = \frac{s^2-1}{2s} = \frac{+121-1}{-22} = -\frac{60}{11}$, and $y = -\frac{60x}{11}$; also $\frac{z}{x} = \frac{s^2-4}{4s} = \frac{+121-4}{-44} = -\frac{117}{44}$, and $z = -\frac{117x}{44}$

In these expressions y and z are both negative quantities, but x is positive, and may take any value we please, without altering the result; let $x=1$, then $y = -\frac{60}{11}$ and $z = -\frac{117}{44}$; let these be brought to the same denominator, then $x = \frac{44}{44}$, $y = \frac{-240}{44}$, $z = -\frac{117}{44}$, or discharging the denominators, and taking y and z for plus terms, we have $x = 44$, $y = 240$, and $z = 117$, which is Euler's answer, as above noticed.

Now $(44)^2+(240)^2 = 59536 = \square$, its root being 244,

$(44)^2+(117)^2 = 15625 = \square$, its root being 125,

and $(240)^2+(117)^2 = 71289 = \square$, its root being 267.

Note 1. If, in the beginning, I had assumed $\frac{y}{x} = \frac{s^2-1}{2s}$, and $\frac{z}{x} = \frac{4s-1}{4s}$, I should have derived the same values for x, y and z as above, and all would have been positive, but the process would have been a little more troublesome.

Note 2. If we assume $\frac{y}{x} = \frac{a^2s^2-b^2}{2ab}$, and $\frac{z}{x} = \frac{c^2s^2-d^2}{2cd}$, we shall, in the foregoing manner, by varying the values of a, b, c, and d, obtain all the answers, which this problem is susceptible of.

4) Note 3. If the problem had been, to find three square numbers, x^2, y^2 and z^2, such that the difference of any two of them should be a square number, the process would have been very different from the foregoing, but of this I shall take no further notice at present.

Another of Bonnycastle's Diophantine Problems is as follows.

To find three square numbers, such that their sum being severally added to their three sides shall make square numbers. Vide Introd. Prob. 22. Pag. 162.

Bonnycastle concludes, that this problem is incapable of an answer in whole numbers, and I believe no three squares can be found, in the whole table of squares, that will give an answer in whole numbers.

The answer in the introduction is $\frac{44181}{62920}$, $\frac{13254}{62920}$ and $\frac{19881}{62920}$, which are the three roots of the required squares.

Without attempting to find the foregoing answer, let us endeavour to find an answer, which shall be expressed in more simple terms.

Let, therefore, $\frac{6x}{5}$, $\frac{8x}{5}$, and $\frac{2}{5}$ represent the roots of our intended squares; then $\frac{36x^2}{25}$, $\frac{64x^2}{25}$ and $\frac{4}{25}$ will be the squares themselves; whence $\frac{36x^2}{25}+\frac{64x^2}{25}+\frac{4}{25}=\frac{100x^2}{25}+\frac{4}{25}=4x^2+\frac{4}{25}$, which is the sum of the squares.

Now, by the conditions of the question,

$$\left.\begin{array}{r} 4x^2+\frac{6x}{5}+\frac{4}{25} \\ 4x^2+\frac{8x}{5}+\frac{4}{25} \\ \text{and } 4x^2+\frac{2}{5}+\frac{4}{25}\text{, or, } 4x^2 \cancel{x} + \frac{14}{25} \end{array}\right\}\text{ are to be }\square^{s}.$$

But $4x^2+\frac{8x}{5}+\frac{4}{25}$, having the form of a square, will always make a square number, let x be of what value it will, it therefore only remains to make $4x^2+\frac{14}{25}$ and $4x^2+\frac{6x}{5}+\frac{4}{25}$ square numbers.

To effect this, let $4x^2+\frac{14}{25}=(2x+r)^2$, or $4x^2+4rx+r^2$, then, by reduction, we have $x=\frac{14-25r^2}{100r}$. Let this value of x be substituted in the other quantity $4x^2+\frac{6x}{5}+\frac{4}{25}$, and it will be transformed into

$$\frac{784+1680r-1200r^2-3000r^3+2500r^4}{10000r^2}.$$ Now

since the denominator of this last expression is a □, it may be thrown away, and the numerator is also divisible by the square 4, whence I derive

$196+420r-300r^2-750r^3+625r^4$, which only remains to be made a square. Let its root be $p-15r+25r^2$; then $196+420r-300r^2-750r^3+625r^4=p^2-30pr+(50p+225)r^2-750r^3+625r^4$. Now $(50p+225)r^2=-300r^2$, because these terms are homologous, and consequently $p=-\frac{21}{2}$, and if we equate the remaining terms which do not cancel one another, we shall have $196+420r=p^2-30pr$; whence $r=+\left(\frac{441}{4}-196\right)\div(420+315)=-\frac{49}{60}$; hence x, or $\frac{14-25r^2}{100r}=\frac{14-\frac{60025}{3600}}{}=\frac{-9625}{-294000}=+\frac{77}{2352}$; and this value of x, being substituted in the assumed roots $\frac{6x}{5}$, $\frac{8x}{5}$ and $\frac{2}{5}$, they become $\frac{462}{11760}$, $\frac{616}{11760}$, and $\frac{4704}{11760}$, but may be reduced to $\frac{33}{840}$, $\frac{44}{840}$, and $\frac{336}{840}$, which is the ans. required.

Suppose we should encumber our last problem with another condition, and make it read thus.

To find three square numbers such, that not only their sum shall make a square number, but if their sides be severally added to the sum of the squares, they shall all make square numbers.

If the question be as above stated, it will evidently be more difficult, than that which we have already solved, because a new condition is now added, namely, the three squares sought, when added together, are to make a square; let us, however, attempt the solution of this problem.

Let x, $\frac{x}{2}$, and $\frac{x}{3}$ represent the roots sought.

Then $x^2+\frac{x^2}{4}+\frac{x^2}{9}=\frac{49x^2}{36}$, which is a □, whence the new condition of the problem will, in every circumstance of x, be satisfied; it remains then, according to the other conditions,

to make $\left.\begin{array}{l}\frac{49x^2}{36}+x\\ \frac{49x^2}{36}+\frac{x}{2}\\ \text{and } \frac{49x^2}{36}+\frac{x}{3}\end{array}\right\}$ square numbers.

Let the square root of $\frac{49x^2}{36}+\frac{x}{3}=\frac{7x}{6}+r$, then will $\frac{49x^2}{36}+\frac{x}{3}=\left(\frac{7x}{6}+r\right)^2=\frac{49x^2}{36}+\frac{7rx}{3}+r^2$, whence, by reduction, we have $x=\frac{3r^2}{1-7r}$. Let this value of x be substituted in the two remaining quantities $\frac{49x^2}{36}+\frac{x}{2}$, and $\frac{49x^2}{36}+x$, and they will become $\frac{441r^4-378r^3+54r^2}{36-504r+1764r^2}$, and $\frac{441r^4-756r^3+108r^2}{36-504r+1764r^2}$. Now since these denominators are squares, they may be thrown away as unnecessary, and the numerators may be divided by the square $9r^2$, whence we derive $49r^2-42r+6=\Box$, and $49r^2-84r+12=\Box$. Make the square root of $49r^2-42r+6=7r-3s$; then will $49r^2-42r+6=49r^2-42sr+9s^2$, whence $r=\frac{3s^2-2}{14s-14}$. Let this value of r, as here expressed in terms of s, be substituted, in the place of r, in the quantity $49r^2-84r+12$, and it will become $\frac{441s^4-3528s^3+5292s^2-2352s+196}{196s^2-392s+196}$, but, as the denominator of this is a square, it may be rejected, and it will only remain then to make

$441s^4-3528s^3+5292s^2-2352s+196$ a square.

Let its square root be $ps^2-84s+14$, then proceeding as before directed, it will be, $p=-63$, and $s=-4$, whence $r=\frac{3s^2-2}{14s-14}=\frac{48-2}{-56-14}=\frac{46}{-70}=+\frac{23}{35}$; and x, or $\frac{3r^2}{1-7r}=\frac{1587}{1225}\div\frac{196}{35}=+\frac{1587}{6860}$; and so the roots sought, viz. x, $\frac{x}{2}$, and $\frac{x}{3}$, do become $\frac{9522}{41160}$, $\frac{4761}{41160}$, and $\frac{3174}{41160}$; and these values will be found to fulfil all the conditions of the problem.

Another of those problems is the following;

Two cube numbers (8 and 1) being given; to find two other cube numbers, whose difference shall be equal to the sum of the given cubes.

See the Introduction before mentioned, Prob. 29. Page 162.

Bonnycastle's answer is $\frac{8000}{343}$ and $\frac{4913}{343}$, which are the cubes required, but this question is capable of various answers, tho' the above may be the most convenient one.

Let $x+r$ and x denote the cube roots sought, then

$x^3+3x^2r+3xr^2+r^3-x^3=9$, by the statement;

or $3x^2r+3xr^2+r^3=9$, and by transposition, &c,

$3rx^2+3r^2x=9-r^3$, but dividing by $3r$, we have

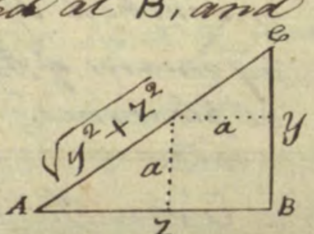
$x^2+rx=\frac{9-r^3}{3r}$, whence, by completing the square, &c,

$x^2+rx+\frac{r^2}{4}=\frac{9-r^3}{3r}+\frac{r^2}{4}=\frac{108r-3r^4}{36r^2}$,
and, by extraction of roots, $x=\frac{-r}{2}+\frac{\sqrt{108r-3r^4}}{6r}$. If, in this quantity, r be taken $=3$, the radical will become a $\square$; but while r remains $=3$, we can obtain no solution of the question; let us therefore change the form of the radical, by making $r=q+3$; this, being substituted, gives $108r-3r^4=-3q^4-36q^3-162q^2-216q+81$; and now the square root of the right hand member of this last equation being required, let said root be represented by $+pq^2+12q-9$; then if we proceed, in all respects, as before explained, p will come out $=17$, and $q=-\frac{111}{73}$, whence r (or $q+3$) $=\frac{108}{73}$.
Having obtained a value for r, we have the assumed roots, to wit $x=-\frac{r}{2}+\frac{\sqrt{108r-3r^4}}{6r}=-\frac{r}{2}+\frac{pq^2+12q-9}{6q+18}=\frac{271}{438}$, and $x+r=\frac{271}{438}+\frac{108}{73}=\frac{919}{438}$; and these values of x and r will be found to answer the conditions of the problem, for $\left(\frac{919}{438}\right)^3-\left(\frac{271}{438}\right)^3=9$, as required.

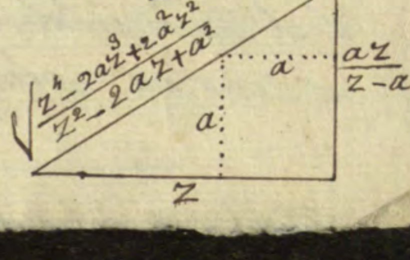
Problem of my own.

Required the exact lengths of the boundary lines of a piece of land lying in the form of a right angled triangle, whose greatest inscribed square shall contain any given number of square rods, say just 25.

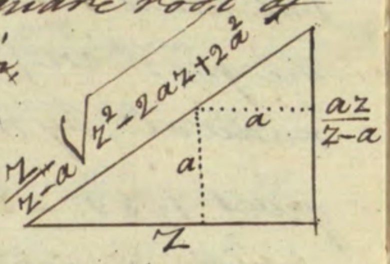
First, let us assume a triangle, right angled at B, and denote its sides, as in the annexed figure;



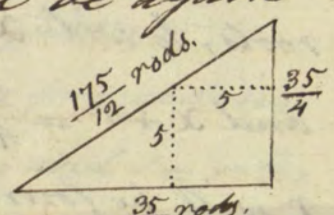
that is to say, a = the side of the given square $=5$; the base $AB=z$; the cathetus $BC=y$, and the Hypothenuse, by a well known theorem, $=\sqrt{y^2+z^2}$.
Next I institute the following proportion, viz.
As $z:y::a:\overline{y-a}$, which, being turned into an equation, gives $\frac{ay}{z}=y-a$, whence, by multiplication, we have $ay=zy-az$, and, by reduction, $y=\frac{az}{z-a}$.
For y I now substitute $\frac{az}{z-a}$, which, by so doing, is transformed into this,



8) In our latter triangle we have but one unknown letter, namely z, but the quantity expressing the length of the hypothenuse may be reduced to a more convenient form by freeing z^2 from the radical sign of the numerator, and extracting the square root of the denominator, it will then be as in the margin.

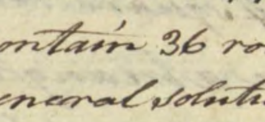


As our hypothenuse contains now only two factors, viz. $\frac{z}{z-a}$, and the radical $\sqrt{z^2-2az+2a^2}$, we may find a value for z, by making the square root of $z^2-2az+2a^2 = z-4a$; then, by involution, we have $z^2-2az+2a^2=(z-4a)^2=z^2-8az+16a^2$, which, reduced, gives $z=\frac{7a}{3}$. Let 5 be now substituted for a, then will z or $\frac{7a}{3}=\frac{35}{3}$, and, by substituting these values of a and z in our triangle above, it will be again transformed into the following, viz.



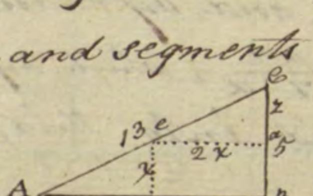
This last figure satisfies all the conditions of the question, for the exact sides of the required triangle are now found, and the greatest square, that can be inscribed therein, contains just 25 square rods, that is, the square 25, as it respects the triangle in question is a maximum, for it touches the 3 sides thereof, and is contiguous to the right angle, as it should be.
If we should take $a=6$, then would $z\left(\frac{7a}{3}\right)=14$, and the sides of the resulting triangle would be 14, $\frac{21}{2}$, and $\frac{35}{2}$, and the inscribed square would contain 36 rods; thus the question, by means of the foregoing general solution, may be varied at pleasure.

Given the ∟ triangle



, to find its greatest inscribed parallelogram, which shall be twice as long as wide.

Here the ▭ in question is to be a maximum, and we may find the exact sides of it in this, or any other example of the kind, in the following manner, viz.
Let the inscribed ▭, and segments of the sides of the triangle be denoted as follow,



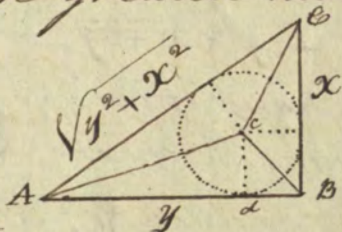
Then, by proportion,

$5 : 12 :: x : \frac{12x}{5} = y = AD$, and again,
$12 : 5 :: 2x : \frac{10x}{12} = z = Ca$; whence the area of the whole triangle becomes, as follows — viz.
the $\square = 2x^2$, and the triangles ADe and $Cae = zx + \frac{yx}{2}$ $= \frac{10x^2}{12} + \frac{12x^2}{10}$; hence $2x^2 + \frac{10x^2}{12} + \frac{12x^2}{10} = 5 \times \frac{12}{2} = 30$, which gives $x = \sqrt{\frac{3600}{484}} = \frac{60}{22} = \frac{30}{11} = 2{,}88+$, being a little more than half the line CB; so of others.

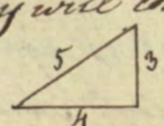
Note 1. If the $\square$ were to be 4, 6, 8, &c, times as long as broad, the sides would be found in the same way, as above shown.

Note 2. If it were required to find the sides of the greatest parallelogram, that could be inscribed in a given $\llcorner$ triangle, the parallelog.m must, in that case, extend half way on each leg; it will then contain just half the triangle.

Let it be required to find the exact sides of a right angled triangle, the diameter of whose greatest inscribed circle shall be exactly 2.



Let the triangle sought be first drawn, and denoted, as in the margin, where $CB = x$, $AB = y$, AC = the square root of the sum of the squares of the two legs, i.e. $= \sqrt{y^2+x^2}$, and let the radius or semidiameter $cd = r = 1$, then will $\frac{r}{2}\sqrt{y^2+x^2} + \frac{ry}{2} + \frac{rx}{2}$ = the area of the whole triangle; but when $r = 1$, as in the present instance, this last equation may be written thus, $\frac{\sqrt{y^2+x^2}+y+x}{2} = \frac{xy}{2}$ = the area; and, by transposition, &c, $\sqrt{y^2+x^2} = xy - y - x$; whence $y^2 + x^2 = (xy-y-x)^2 = x^2y^2 - 2xy^2 + y^2 - 2x^2y + 2xy + x^2$, or $x^2y^2 - 2xy^2 - 2x^2y + 2xy = 0$. This last, divided by xy, gives $xy - 2y - 2x + 2 = 0$, consequently $y = \frac{2x-2}{x-2}$, in which x may be taken at pleasure, provided it be greater than 2.
I also find by another process, which I shall not stop here to explain, that it will suffice to write $y = \frac{x^2 - x + x\sqrt{1-2x+x^2}}{x^2-2x}$ in which expression x may be any number greater than 2.
In the first formula, if x be taken $= 3$, y will come out $= 4$, and the triangle sought will become

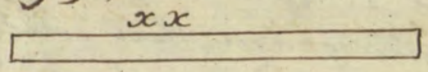


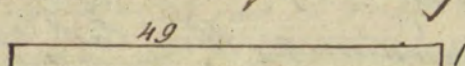
, the diameter of whose greatest inscribed circle will be found to be 2, so of others.

Note. If the Diam.r is required to be of any length other than two, the sides may be determined, by means of the foregoing formulas, by simple multiplication, or division only.

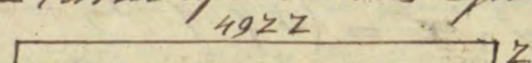
The following question I caused to be inserted in Mann's Dover paper, but received no answer.

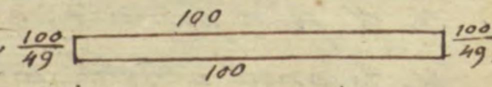
" It is required to find the exact sides of a right angled parallelogram, which shall be longer than it's broad, and, at the same time, each of its sides shall be a square number; also its perimeter and area shall each be a square number; moreover said perimeter and area shall be equal to each other,,

Let the two different sides of the parallelogram sought be first represented by xx and yy, it will then be in the similitude of this figure, 

One condition of the problem is, that the whole perimeter $2xx+2yy$ must be a square number. To fulfil this, let the square root of $2xx+2yy = x+3y$, then will $\overline{x+3y} \times \overline{x+3y}$, or $xx+6yx+9yy = 2xx+2yy$, whence by transposition, &c, we have $xx-6yx = 7yy$; and by completing the square, $xx-6yx+9yy = 7yy+9yy = 16yy$; hence by evolution, &c, $x = 3y+\overline{16yy}|^{\frac{1}{2}} = 3y+4y = 7y$; whence, by taking $y=1$, we have $x = (7y) = 7$, also $xx = 7\times 7 = 49$, and $yy = 1$. Now if we substitute these numeral values for xx and yy in the above parallelog.^{m}, it will be changed into this, 

, the perimeter and area of which is now each a square number, but the perimeter is not equal to the area, as it ought to be by the question.

This last condition, however, has place, if we multiply each side of the latter parallelog.^{m} by zz, and afterwards equate the proper quantities, thus, 

of which the perimeter, $100zz$, is still a square N.^{o}, and the area $49z^{4}$ is a square N.^{o} also; and to make these two quantities equal to each other I equate them thus, $100zz = 49z^{4}$; then dividing each side of the last equation by zz common, there results $100 = 49zz$, which dividing again by 49, gives $zz = \frac{100}{49}$. This value of zz I now substitute in the last parallelog.^{m}, and it is again changed into, 

Now this last parallelog.^{m} satisfies all the conditions, as required; for it is right angled, — it is longer than broad, — each of its four sides is a square N.^{o} — moreover, its perimeter $\frac{10000}{49}$ is a square N.^{o}, — its area $\frac{10000}{49}$ is a square N.^{o}, and lastly its perimeter and area are equal to each other.

The following question was taken out of the Farmer's Almanack for the present year.

A man puts out 100 dollars, at compound interest, until it shall treble the sum; and the rate per cent. is equal to the years, — how much per cent. and what time is required?

Let $R =$ the principal and interest, or amount of one dollar for one year, and let t denote the time or number of years, which the money remains at interest; then will $100 \times R^t = 300$ dol. by the question; wherefore $R^t = \frac{300}{100} = 3$; but $R^t = 3$ is an exponential equation, because one of the unknown letters, viz. t, enters the exponent, and because the labour of finding the approximate values of R and t, by means of the natural numbers, would be very great, we must, therefore, have recourse to logarithms. Thus, the log. of 3 is 0,477121+, and the log. of R × by t must be the same, viz. = 0,477121+. Now I soon find, by trial, that the rate of interest will be between 10 and 11 per cent. but nearer 11 than 10, and consequently the number of years will be somewhere between 10 and 11, for the years must be equal to the rate per cent. Let us first assume R, or the amount of one dollar for one year = 1,107, then the time t will be 10,7 years.

Now 1,107 = Log. 0,044148, and 0,044148 × 10,7 the time = Log. 0,4723836, but this Log. is less than the Log. of 3; that is, 0,4771212

subtract 0,4723836

= 0,0047376, which is an error; wherefore let us again suppose the rate of interest or amount of one dollar as above to be 1,108, then the time will be 10,8 years.

The Log. of 1,108 = $\overset{\text{Log.}}{0,044540}$, and 0,044540 × 10,8 = 0,481032, but 0,481032

subtract 0,477121

= 0,003911; hence 1,108 is too great.

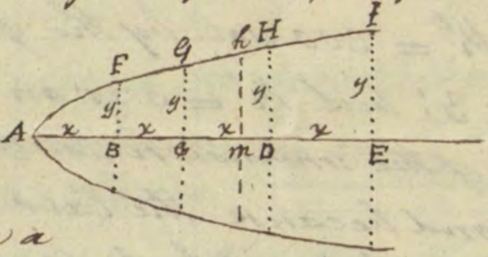
Now − ,0047376

add + ,003911

= ,0086486 = the sum of the errors; wherefore, as ,0086486 : 1 :: $\overset{\text{first error.}}{,0047376}$: $\overset{\text{the correction.}}{,000548}$, and, consequently, 1,107 + ,000548, or 1,107548 is = R, the ratio, extremely near, so that the rate per cent. is 10,7548, or a little more than $10\frac{3}{4}$, and the number of years is also 10,7548. This answer is as near, as a table of Logs. extending to 6 places of decimals, will give these numbers.

12) Let it be required to find whereabouts a stick of timber in the shape of a wedge would poise, or make a balance, if laid across another stick, which in this case might be called its fulcrum.

This question appertains to Fluxions, therefore let us first suppose the triangular edge, or side of the wedgelike solid in question, to be turned into a parabolic superficies, then it will resemble, and possess the properties of, the annexed figure.

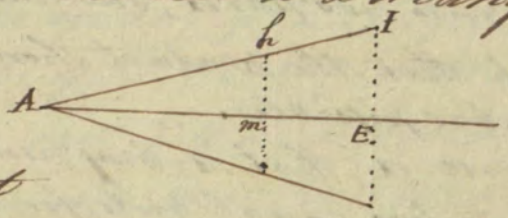


The Center of Gravity, or place where this superficial parabola will make a balance, must be somewhere on the line AE, because the parts on each side of that line will balance one another. The lines BF, CG, DH, &c, are ordinates rightly applied; and the lines AB, or AC, or AD, &c, are abscissas corresponding with those ordinates. Now if the ordinates be denoted by y, and the abscissas by x, it will then be,
as $y^2 : x :: a : 1$, and, converting this analogy into an equation, we have $y^2 = ax$, universally; this is called the equation of the curve. But if $y^2 = ax$, then $y = (ax)^{\frac{1}{2}}$; but, for convenience sake, we may make $y = (a^2x)^{\frac{1}{2}} = ax^{\frac{1}{2}}$, or y may $= ax^n$. —— Suppose we assume $y = ax^n$, as the equation of the parabola, in order to find its center of gravity; then, according to the nature of fluxions, $yx\dot{x} = ax^{n+1}\dot{x}$, (the fluxion $\dot{x}$ of these quantities being distinguished by a dot over it), and the fluent of the right hand member of the last equation above is $\frac{ax^{n+2}}{n+2}$; also since $y\dot{x} = ax^n\dot{x}$, its fluent is $\frac{ax^{n+1}}{n+1}$, hence the length of the line Am will show the distance to be measured from the vertex A to the center of gravity which is situated at the point m, in which case x = the whole line AE. Or x may have any other length, in which case the locus or place of m will vary accordingly. But to find the numeral values of x and Am, I say,

$$Am = \frac{ax^{n+2}}{n+2} \times \frac{n+1}{ax^{n+1}} = \frac{n+1}{n+2} \times x.$$

If $n = \frac{1}{2}$, then $y = ax^{\frac{1}{2}}$, and $y^2 = a^2x$, which is the common parabola; hence Am (or $\frac{n+1}{n+2} \times x$) $= \frac{3}{5}x$, where x may be taken ad libitum, (or of what value we please) and the length Am will be three fifths of the same.

But, if we assume $n = 1$, then y (or ax^n) becomes $= ax$, and the figure is reduced to a triangle, as here represented,



In which case Am, or $\frac{n+1}{n+2} \times x = \frac{2}{3} x$. Hence it follows, that if a stick of timber, or any other solid, in the shape of a wedge, be laid across another stick, so as to make an equipoise, the length of the two ends, thus balanced, will be in the ratio of 2 to 1, that is, the smaller end will be twice as long as the other.

Suppose it were required to determine the place where any conical or pyramidal stick of timber, or other solid of the kind, would, as before stated, form a balance or equipoise, if laid across another stick.

Here, as before, if the proposed solid rests undisturbed, and its center of gravity be supported, it can have no tendency to fall. — A Cone is a solid generated by the revolution of a triangle about the axis, which remains fixed; but let us first consider a paraboloid, which is a solid generated by the revolution of a parabola about its axis. The equation of the parabola, was, as above noticed, $y = ax^n$; but that of the paraboloid will be the square of this, that is, it will be $y^2 = a^2 x^{2n}$, and therefore $y^2 x \dot{x} = a^2 x^{2n+1} \dot{x}$, whose fluent is $\frac{a^2 x^{2n+2}}{2n+2}$, and $y^2 \dot{x} = a^2 x^{2n} \dot{x}$; whose fluent is $\frac{a^2 x^{2n+1}}{2n+1}$; hence

$$Am = \frac{a^2 x^{2n+2}}{2n+2} \times \frac{2n+1}{a^2 x^{2n+1}} = \frac{2n+1}{2n+2} \times x.$$

If $n = \frac{1}{2}$, the solid becomes a paraboloid, and in that case $Am = \frac{2}{3} x$; but if n be taken $= 1$, then $y = ax$, and the solid becomes a perfect cone; and Am, or the distance to be measured from the vertex of the triangle on the line of the abscissas, will now be $= \frac{3}{4} x$, for in this case $\frac{2n+1}{2n+2} \times x = \frac{3}{4} x$.

Note. Since $y = ax$, as above, it may be observed, generally, that the locus of every ~~equa~~ equation, in which the co-ordinates x and y are in separate terms, and do not rise above the first power or dimension, will, necessarily, be a straight line, for every such equation may be brought to the form $x = \frac{y}{a} \pm b$.

 Suppose it were required to divide any given number (a) into two parts, such that, if those parts be multiplied into each other, the product should be a maximum, or greatest number possible.

In this case a, that is, any number whatever, should be divided into two equal parts, for then their rectangle will make a greater number, than that of any two unequal parts into which a may be divided; — and if the parts be more in number than two, their continued product will be greatest, if those parts be equal.

Let it now be required to divide a given number (a) into two parts, such that, if the first part be multiplied into the square of the other, the product shall be a maximum. Here let x and y be the two unknown parts into which the number (a) is to be divided, then will $x+y=a$, and, by the statement, xy^2 is to be the maximum; now the fluxion of each of these $=0$; the former, because it is constant; the latter, because it is a maximum; wherefore $\dot{x}+\dot{y}=0$, and so likewise $y^2\dot{x}+2xy\dot{y}=0$; hence, from the two last equations, we derive $\dot{x}=-\dot{y}$, and $\dot{x}=-\frac{2xy\dot{y}}{y^2}=-\frac{2x\dot{y}}{y}$; hence $-\dot{y}=-\frac{2x\dot{y}}{y}$, or $-y\dot{y}=-2x\dot{y}$, and $y=2x$.

Suppose a above $=6$, then the parts are $x=2$, and $2x$ or $y=4$, and $xy^2=2\times4\times4=32$, which is the maximum req.^d

If, in the above example, it had been required to determine the values of x and y, when xy^3 is to be the maximum; in that case if $a=6$, x will $=\frac{3}{2}$, and $y=\frac{9}{2}$, and $xy^3=\frac{2187}{16}=136\frac{11}{16}=$ a maximum.

And, in general, if $x^m y^n$ is to be a maximum, then will $x=\frac{ma}{m+n}$, and $y=\frac{na}{m+n}$, where a, m and n may be valued at pleasure; if $m=n$, the parts will be equal.

Suppose it were required to divide any given number (a) into three parts, x, y and z, so that the first part multiplied by the square of the second, and that product again by the cube of the third should produce a maximum. — In this case xy^2z^3 is to be the maximum, and the parts x, y and z must have some certain, determinate values to answer the conditions; we need not, at present, inquire what any of those values may be, it suffices to know, that to each part its own determinate value may at last be assigned, and that it cannot have more values than one.

This being premised, let us now suppose, that this true value of one of the parts, say of y, whatever it may be, remains constant, while the values of x and z vary, till they answer the conditions, then we shall have $\dot{x}+\dot{y}=0$, and $z^3\dot{x}+3xz^2\dot{z}=0$; hence $\dot{x}=-\dot{z}=-\frac{3xz^2\dot{z}}{z^3}=-\frac{3x\dot{z}}{z}$, ergo $z=3x$.

Now let us suppose the value of z to remain constant, and x and y to vary, till they satisfy the conditions; then $\dot{x}+\dot{y}=0$, and $y^2\dot{x}+2xy\dot{y}=0$; hence $\dot{x}=-\dot{y}=-\frac{2xy\dot{y}}{y^2}=-\frac{2x\dot{y}}{y}$, and $-y\dot{y}=-2x\dot{y}$, ergo $y=2x$.

Consequently, by substituting these values for y and z, it appears that a is to be divided into the three parts, x, $2x$ and $3x$, or into $\frac{1}{6}a$, $\frac{1}{3}a$, and $\frac{1}{2}a$, when it is required, that xy^2z^3 should be a maximum.

If $a=6$, as before, then x will $=1$, $y=2$, and $z=3$, and $1\times(2)^2\times(3)^3=108$, the maximum sought.

Note. According to the nature of things there can be no minimum required in questions of this kind, because the supposed minimum, if it could reach 0, would not stop there, but run into negative numbers infinitely, if such numbers were admissible.

Of Logarithms.

Many curious things may be effected by Logarithms, which would be extremely laborious and difficult, if not altogether impracticable, by the natural numbers. I have not time to enlarge upon this subject, but a few instances here follow.

Let it be required to find the ,0726th root of ,2603, or to determine the value of $\overline{,2603}|^{\frac{1}{,0726}}$.

Numb. ,2603 = Log. −1,41547.

To the characteristic (or index) −1 of this log. I add ,452, in order to render it divisible by ,0726.

Then ,0726) −1,4520 (−20 for the index.
−1 452
. . . . 0

But adding what I borrowed, viz ,452 to the mantissa of the dividend, we have ,41547
,452
,86747 for a dividend.

Therefore, by division, ,0726) ,86747 (11,94862, the quotient.

To the quotient 11,94862
I add the index −20
& thence obtain the Log. −9,94862, which is equal to the natural number, 000000000 88842 = the root required.

Note. This root is equivalent to the 13,7741 th power of ,2603, for $\frac{1}{,0726}$ = 13,7741, &c. and so of others.

Let it now be required to find, by a contrary process, the 13,7741th power of the aforesaid natural number, ,2603.

Log.
,2603 = −1,41547
multiply by the index, 13,7741
−1,41547
−3,66188
−5,90829
−5,90829
−2, | 24647
−1, 4 | 1547
−9,948625327
= nat. numb. 000000000 88842.
as above shewn.

Note. The first figure in each line of the partial products, viz. −1, −3, −5, &c, is negative, because it arises from the multiplication of −1 the index of the multiplicand. In adding up the columns of the product I proceed as in common arithmetick, (taking care however to subtract the negative figures as I go along) till I get to the 9th column $-\frac{5}{2}{}_{1}$, which I add up thus; I say +1 I carry to 1 makes 2, and 2 make 4, and −5 make −1, which is negative, wherefore to make this positive I borrow +10, and so have −1+10 = +9, which I set down, and then the mantissa, or decimal part, is completed, for it now has 9 places of decimals, as it ought to have, there being 9 places in the multiplicand and multiplier.

All the remaining figures of the partial products I consider as belonging to the index, namely −1, $\overset{-2}{4}$; these = −12 + 4, but as I have −1 to carry from the column on the right, it makes −13+4 = −9 for the index, which I set down as above. Now the Log. −9,948625327 is the same nearly as the Log. −9,94862 above, which shows, that both the foregoing operations are correct. Thus any fractional or other root or power of any number whatsoever may be readily determined by Logarithms.