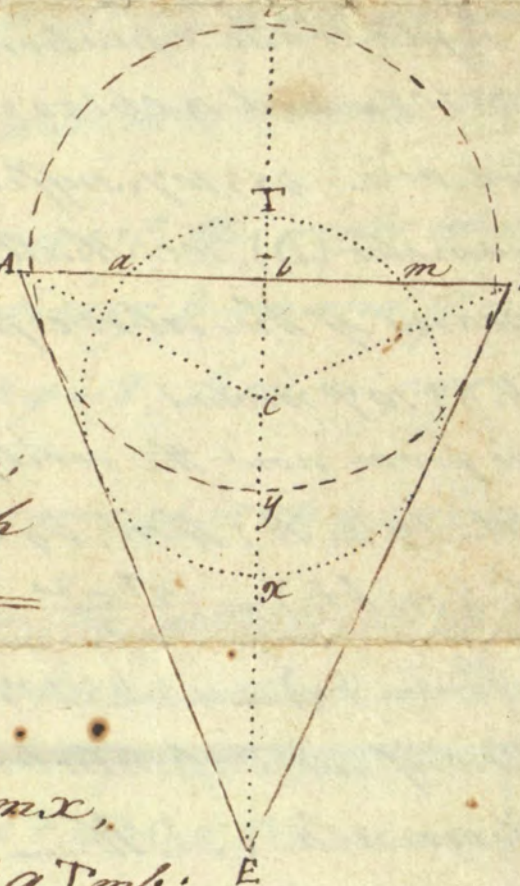
Question. There is a conical glass, 6 inches in depth, and 5 inches wide at the top within. Now if this glass be filled three fourths full of water, what will be the diameter of a ball or sphere which, being dropped into the glass, will throw out just half a pint of the liquid, wine measure; also, how far into the glass will the ball sink?
Note, if this question be susceptible of two answers, what are they, and what will be the method of finding them?



Let the glass, &c. be represented as in the margin.

Then $5 \times 5 \times 0.785398+ \times 2 = 39.2699$ cub. in. = the content of the conical glass, very near.

$\frac{1}{4}$ of this $= 9.817475$ cub. in.

Also $\frac{1}{2}$ a pint, wine mea. $= 14.4375$ cub. in.;

Hence that portion of the ball which enters the cone will $= 9.817475 + 14.4375 = 24.254975$ cub. in.

Now the line AE (or BE) $= 6.5$ in.

But let $D =$ the diam.r of the ball aTmx, and $h = bT$, the height of the segment aTmb;

then will the solidity of the frustrum or segment aTmb $=$ $(3Dhh - 2hhh) \times 0.5235986+$ (see Ward's Arith. of infinites).

Now since $.5235986\,D^3 =$ the solidity of the ball, we have the equation $(D^3 - 3Dhh + 2hhh) \times .5235986 = 24.254975 =$ that part of the ball which falls within the cone.

Let the radius (or $\frac{1}{2}$ diam. of the ball) $= R$, then will $R - h = bc$, and we shall derive $\frac{13}{2}R + \frac{5R - 5h}{2} = 15 =$ the area of the triangle ABE. This last equation, reduced, gives $h = \frac{18R - 30}{5} = \frac{9D - 30}{5}$, by writing D instead of $2R$.

Let now $\frac{9D-30}{5}$ be substituted for h in the equation $(D^3 - 3Dhh + 2hhh) \times .5235986 = 24.254975$, and it will become $(368D^3 - 6480D^2 + 35100D - 54000) \times \frac{.5235986}{125} = 24.254975$,

which may be reduced to $92D^3 - 1620D^2 + 8775D = 14947.612672$.
This last is a cubic equation that may be resolved by divers of the approximating rules; — thus I find D, the diameter of the sphere sought, to be 3.6961 + inches; consequently the height (h) of the segment $aTmb = \frac{9D-30}{5} = \frac{9 \times 3.6961+}{5} =$ .65298 inches. Now if we divide the equation

$92D^3 - 1620D^2 + 8775D - 14947.612672 = 0$, by $D - 3.6961$, the quotient will be $92D^2 - 1279.9588D + 4044.14427932 = 0$, which is a quadratic equation, and its two roots, or values of D, are 4.85107+ and 9.0615256+; and since the smaller of these two, viz. 4.85107+ suffices, in this case, for a value of the diameter (D), we shall have two answers to the question by means of the above process, that is D = 3.6961, and 4.85107 inches.

To prove now the verity of the solution D = 4.85107, let $\underline{by} = H$; then, by taking D = 4.85107, we shall have h (or bt) $= \frac{9D-30}{5} = 2.731926$ in. and

$H\ (= by) = 4.85107 - 2.731926 = 2.119156$ in

Now if we substitute these values of D and H in the formula $(3DH^2 - 2H^3) \times .5235986$, we shall find that it will amount to 24.254975 cub. in. (very nearly), as it ought to do.

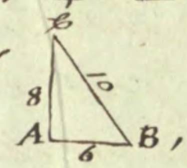
Note 1. The lesser ball sinks into the glass $3.6961 - 0.65298 = 3.04312$ in., and the greater one 2.119156 inches.

Note 2. The root, or value of D found to be 9.0615256 is unfit for our purpose, and therefore no other answer, but the two above specified, can be given to the foregoing problem.

Ques. 2. It is required to find the sides of a plane triangle, being such that its area shall be a perfect integral square, and its perimeter equal to its area.

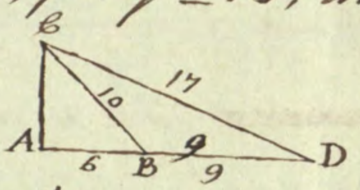
The quantities r^2+s^2, r^2-s^2, and $2rs$, by varying the values of r and s, will represent the integral (or fractional) sides of any right angled triangle whatever, provided r be greater than s.

Suppose $r=3$, and $s=1$, then $r^2+s^2=10$, $r^2-s^2=8$, and $2rs=6$; whence our triangle becomes



, whose area is $6\times4=24$.

Again let p^2+q^2, p^2-q^2 and $2pq$ represent the sides of a right angled triangle, then by taking $p=4$ and $q=1$, we have $p^2+q^2=17$, $p^2-q^2=15$, and $2pq=8$; hence our triangle becomes



, whose area is $15\times4=60$.

Subduct now the first triangle from the second, and there is left the triangle BCD, as above, whose area is $60-24=36$, which is a perfect integral square. The perimeter also is $10+9+17=36$; hence the conditions of the problem are all satisfied.

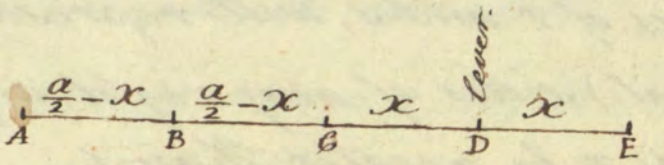
<u>Note</u>. The triangle BCD is an oblique angled scalene triangle.

Question in Mr. Leavitt's Almanach for 1833.

Suppose there is a stick of timber 15 feet in length, and of an equal bigness in every part, and three men should undertake to carry it; one to take hold of one end, and the other two to carry their end on a lever. How far from either end of the stick must the lever be placed, so that each man may carry just a third part of the weight of the timber, if the two men who lift together stand at equal distances from the stick?

(turn over)

Without attending particularly to the numbers mentioned in this question, — let $\underline{a}$ denote, generally, the whole length (in feet, &c.) of a stick of timber of the proposed sort; also let x = the distance (in feet) of the lever from the fore end of the stick, as in the following representation.



Then since x feet of timber before the lever will balance x feet behind it, the two men, by this position of the lever, will carry $2x$ feet of the stick.
But now to find how much they will carry of the remaining part of the stick, I institute this proportion, viz. as $AD : AC :: AB$ (or BC) : a fourth proportional;
or, in species, as $a-x : a-2x :: \frac{a}{2}-x : \frac{a^2-4ax+4x^2}{2\times(a-x)}$ = the quantity to be added to $2x$ above mentioned.
Hence $2x + \frac{a^2-4ax+4x^2}{2a-2x} = \frac{a^2}{2a-2x}$ represents, generally, the number of feet carried by the two men with the lever. Suppose now the two men to carry b feet of the timber, then $\frac{a^2}{2a-2x} = b$, and $a^2 = 2ab - 2bx$, whence $x = \frac{2ab-a^2}{2b}$, where $\underline{a}$ may denote any number whatever, and b any number not greater than $\underline{a}$, nor less than $\frac{a}{2}$.
If $\underline{a}$, in the formula, be taken = 15 feet, and $b = 10$, (or $\frac{2}{3}$ of the whole stick), then will the corresponding value of x (or $\frac{2ab-a^2}{2b} = \frac{300-225}{20} = \frac{75}{20} = 3\frac{3}{4}$ feet.
Hence the lever must be placed 3 feet, 9 in. from the end of the stick, if the three men are to carry an equal weight each. Note, if the lever be placed 5 feet from the end of the stick, the two men will carry just $\frac{3}{4}$ of the whole weight, nothing being allowed for the weight of the lever.